Quantization of Teichmüller spaces and the quantum dilogarithm

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Abstract

The Teichmüller space of punctured surfaces with the Weil–Petersson symplectic structure and the action of the mapping class group is realized as the Hamiltonian reduction of a finite dimensional symplectic space where the mapping class group acts by symplectic rational transformations. Upon quantization the corresponding (projective) representation of the mapping class group is generated by the quantum dilogarithms.

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Introduction

Despite the progress in understanding and solving the quantum Chern–Simons field theory with a compact gauge group (Witten 1989) much remains unclear in the case of non-compact groups. Among physical motivations for the study of the latter case are the interpretation of $2+1$–dimensional gravity as the Chern–Simons theory with a non-compact gauge group (Witten 1988/89) and identification of the Hilbert space of physical states of the Chern–Simons theory with $SL(2,\mathbb{R})$ gauge group with the space of Virasoro conformal blocks (Verlinde 1990). The major mathematical motivation is provided by the possibility of constructing new topological three–manifold invariants (Witten 1988/89, 1989).

The purpose of this paper is to quantize the Teichmüller space of punctured surfaces with the Weil–Petersson simplectic structure. This space can be identified with (a part of) the phase space of the $SL(2,\mathbb{R})$ Chern–Simons theory (Verlinde 1990; Witten 1990). We start from the Penner parameterization of the (decorated) Teichmüller space (Penner 1987), where the mapping class group is realized explicitly through rational transformations generated by compositions of the elementary Ptolemy transformation. The latter transformation is canonical and the quantum dilogarithm (Faddeev and Kashaev 1992; Faddeev 1995) implements this transformation on the quantum level. Our approach is similar to the combinatorial quantization of the Chern–Simons theory with compact gauge groups (Fock, Rosly 1992; Alekseev *et al.* 1995; Buffenoir, Roche 1996).

The paper is organized as follows. In Sect. 1 first the results of Penner, to be used in the paper, are formulated using the category language, and then the Teichmüller space is described as the Hamiltonian reduction of a finite dimensional symplectic space. The quantization is performed in Sect. 2. All the proofs are omitted for they are straightforward verifications.

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1 The phase space

1.1 Notation

In this paper $\mathbb{R}_+$ denotes the set of *strictly* positive real numbers with a Lie group structure given by the multiplication. If X is a finite set, $\mathbb{R}_+^X := \{f: X \to \mathbb{R}_+\}$ denotes the space of positive functions on X which is an Abelian group w.r.t. the pointwise multiplication. The corresponding factor–group w.r.t. the subgroup of constant functions is denoted

$$P\mathbb{R}_+^X := \mathbb{R}_+^X / \mathbb{R}_+.$$

Each element $x \in X$ is identified with a coordinate function on $\mathbb{R}_+^X$, and projective coordinate function in $P\mathbb{R}_+^X$:

$$X \ni x: \mathbb{R}_+^X \ni f \mapsto x(f) := f(x) \in \mathbb{R}_+.$$

For set Y groupoid $\mathfrak{G}_Y$ is defined by $\mathrm{Ob}\mathfrak{G}_Y := Y$, each set of morphisms consisting of only one element,

$$\mathrm{Mor}(x,y) := \{f\colon y \to x\} = \{x \cdot y\}, \quad \forall x, y \in Y.$$

$\mathfrak{P}$ denotes a category with $\mathrm{Ob}\mathfrak{P} := \{P(M,G,\pi,\alpha)\}$, where $P(M,G,\pi,\alpha)$ is a principal bundle with total space P, base manifold M, structure group G, projection $\pi\colon P \to M$, and structure group invariant closed two-form α in P. The set $\mathrm{Mor}(P_1,P_2)$, where $P_i := P_i(M_i,G_i,\pi_i,\alpha_i)$, $i = 1,2$, consists of principal bundle morphisms $f\colon P_2 \to P_1$ such that $f^*\alpha_1 = \alpha_2$.

1.2 The decorated Teichmüller space

Let Σ be a closed oriented surface of genus g with $s > 0$ removed points (punctures) $P_1, \ldots, P_s$, where

$$2g - 2 + s > 0.$$

Denote the set of punctures

$$V := \{P_1, \ldots, P_s\}.$$

Let $\mathcal{T}_\Sigma$ be the Teichmüller space of marked conformal types of hyperbolic metrics on Σ, and $\tilde{\mathcal{T}}_\Sigma$, the decorated Teichmüller space, which is a principal $\mathbb{R}^s_+$ foliated fibration $\phi\colon \tilde{\mathcal{T}}_\Sigma \to \mathcal{T}_\Sigma$, where the fiber over a point of $\mathcal{T}_\Sigma$ is the space of all horocycles about the punctures of Σ.

Definition 1 *A homotopy class of a path, running between P_i and P_j, is called* ideal arc (i.a.). *A set of ideal arcs, obtained by taking family X of disjointly embedded simple arcs in Σ running between punctures and subject to the condition that each component of $\Sigma \setminus X$ is a triangle, is called* ideal triangulation (i.t.). *The set of all i.t. on Σ is denoted Δ_Σ.*

Suppose that $a, b, c, d, e \in \tau \in \Delta_\Sigma$, are such that $\{a,b,e\}$ and $\{c,d,e\}$ bound distinct triangles. The operation of changing i.t. τ into i.t. τ^e, which consists in replacing i.a. e by a complementary i.a. e' such that triangles, bounded by $\{a,b,e\}$ and $\{c,d,e\}$, are replaced by triangles, bounded by $\{b,c,e'\}$ and $\{d,a,e'\}$, is called *elementary move* along i.a. e, see Fig. 1.

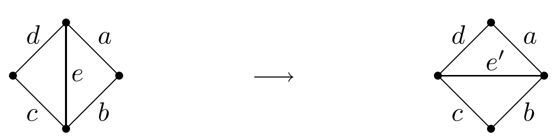

Figure 1:

To each $\tau \in \Delta_\Sigma$ associate object

$$R(\tau) := \mathbb{R}^\tau_+(\mathbb{R}^\tau_+/\mathbb{R}^V_+, \mathbb{R}^V_+, \pi_{R(\tau)}, \alpha_\tau) \in \mathrm{Ob}\mathfrak{P},$$

where the free action of the structure group is defined by

$$\mathbb{R}_+^V \ni f\colon \mathbb{R}_+^\tau \to \mathbb{R}_+^\tau, \quad f^*(c) := f(c_0)f(c_1)c, \quad c \in \tau, \tag{1}$$

with c_0 and c_1 being the punctures connected by i.a. c;

$$\alpha_\tau := \sum d\ln a \wedge d\ln b + d\ln b \wedge d\ln c + d\ln c \wedge d\ln a, \tag{2}$$

where summation is taken over all triangles with a, b, c being edges of a triangle taken in the clockwise order w.r.t. the orientation of Σ. Define also

$$R(\star) := \tilde{\mathcal{T}}_\Sigma(\mathcal{T}_\Sigma, \mathbb{R}_+^s, \phi, \phi^*\omega_{WP}/2) \in \mathrm{Ob}\mathfrak{P},$$

where ω_{WP} is the Weil–Petersson Kähler form in $\mathcal{T}_\Sigma$.

For each $\tau \in \Delta_\Sigma$ define mapping

$$R(\tau\cdot\star)\colon R(\star) \to R(\tau), \quad R(\tau\cdot\star)(h)\colon \tau \ni c \mapsto \sqrt{2}e^{\delta_h(c)/2} \in \mathbb{R}_+, \quad h \in \tilde{\mathcal{T}}_\Sigma, \tag{3}$$

where real number $\delta_h(c)$ is the signed $\phi(h)$-Poincaré distance between the horocycles about the puncture(s) connected by c along the geodesic isotopic to c.

For a pair of i.t τ and τ^e, connected by the elementary move along i.a. e, associate mapping

$$R(\tau^e\cdot\tau)\colon R(\tau) \to R(\tau^e), \quad R(\tau^e\cdot\tau)^*(x) := \begin{cases} (ac+bd)/e & \text{if } x = e', \\ x & \text{otherwise;} \end{cases} \tag{4}$$

see Fig. 1 for the notation of i.a.

Theorem 1 (Penner 1987a, 1987b) *R extends to a unique covariant functor from $\mathfrak{G}_{\Delta_\Sigma\cup\star}$ into $\mathfrak{P}$.*

The mapping class group M_Σ of Σ naturally acts both in $\tilde{\mathcal{T}}_\Sigma$ and Δ_Σ, each i.t. as a set being mapped to its image in Δ_Σ. To each $m \in M_\Sigma$ and $x \in \Delta_\Sigma \cup \star$ associate morphism $\mathfrak{r}_m(x) \in \mathrm{Mor}(R(x), R(x))$ such that

$$R(x) \ni f \mapsto \mathfrak{r}_m(x)(f) := \begin{cases} R(x\cdot m(x))(f\circ m^{-1}) & x \in \Delta_\Sigma; \\ m(f) & x = \star. \end{cases}$$

Theorem 2 (Penner 1987a) *Mapping $m \mapsto \mathfrak{r}_m$ is a group homomorphism from M_Σ into functorial isomorphisms from R to R.*

1.3 Teichmüller space as the phase space of a constraint system

Definition 2 *An i.t. with a choice of a distinguished corner for each triangle is called* decorated ideal triangulation (d.i.t.). *The set of all d.i.t. of Σ is denoted $\tilde{\Delta}_\Sigma$.*

There is a natural projection functor

$$E\colon \mathfrak{G}_{\tilde{\Delta}_\Sigma} \to \mathfrak{G}_{\Delta_\Sigma}.$$

Denote by $\dot\tau$ the corresponding to $\tau \in \tilde{\Delta}_\Sigma$ set of triangles on Σ with distinguished corners.

D.i.t. τ_t, obtained from d.i.t τ by a change of the distinguished corner of triangle t as indicated in Fig. 2, is said to be obtained from τ by the *elementary*

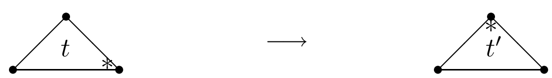

Figure 2:

change of decoration in triangle t. D.i.t. τ^e, obtained from d.i.t. τ by the elementary move along i.a. e, where distingushed corners of the surrounding triangles are such as indicated in Fig. 3, is said to be obtained from τ by the

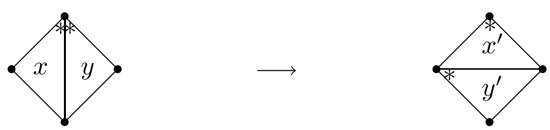

Figure 3:

decorated elementary move along i.a. e.

Denote by k_t and k^t, $k = 0, 1, 2$, punctures and i.a., respectively, on the boundary of triangle t with a distinguished corner in accordance with Fig. 4.

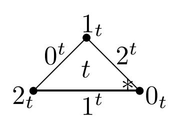

Figure 4:

With each d.i.t. τ associate object

$$S(\tau) := (\mathbb{R}_+^{\dot\tau} \times \mathbb{R}_+^{\dot\tau})((\mathbb{R}_+^{\dot\tau} \times \mathbb{R}_+^{\dot\tau})/P\mathbb{R}_+^V, P\mathbb{R}_+^V, \pi_{S(\tau)}, \beta_\tau) \in \mathrm{Ob}\mathfrak{P}, \tag{5}$$

where the free structure group action is defined by

$$P\mathbb{R}_+^V \ni f\colon S(\tau) \to S(\tau), \quad f^*(\mathbf{t}) := (t_1 f(1_t)/f(2_t), t_2 f(1_t)/f(0_t)); \tag{6}$$

$$\beta_\tau := \sum_{t\in\dot\tau} d\ln t_1 \wedge d\ln t_2; \tag{7}$$

the coordinate functions on $S(\tau)$ used here are defined as follows,

$$\mathbf{t} := (t_1, t_2), \quad t_i := t \circ \mathfrak{pr}_i, \quad t \in \dot{\tau}, \quad \mathfrak{pr}_i \colon S(\tau) \ni (f_1, f_2) \mapsto f_i \in \mathbb{R}_+^{\dot{\tau}}.$$

$S(\tau)$ is a Lie group w.r.t the product of the group structures on $\mathbb{R}_+^{\dot{\tau}}$ factors.

For d.i.t. τ and τ_t, connected by the elementary change of decoration in triangle t, see Fig. 2, associate mapping

$$S(\tau_t \cdot \tau) \colon S(\tau) \to S(\tau_t), \quad S(\tau_t \cdot \tau)^*(\mathbf{x}) := \begin{cases} (t_2/t_1, 1/t_1) & \text{if } x = t'; \\ \mathbf{x} & \text{otherwise.} \end{cases} \tag{8}$$

For d.i.t. τ and τ^e, connected by the decorated elementary move along i.a. e depicted in Fig. 3, associate mapping

$$S(\tau^e \cdot \tau) \colon S(\tau) \to S(\tau^e), \quad S(\tau^e \cdot \tau)^*(\mathbf{t}) := \begin{cases} \mathbf{x} \bullet \mathbf{y} & t = x'; \\ \mathbf{x} * \mathbf{y} & t = y'; \\ \mathbf{t} & \text{otherwise;} \end{cases} \tag{9}$$

where

$$\mathbf{x} \bullet \mathbf{y} := (x_1 y_1, x_1 y_2 + x_2), \quad \mathbf{x} * \mathbf{y} := (y_1 x_2 (x_1 y_2 + x_2)^{-1}, y_2 (x_1 y_2 + x_2)^{-1}). \tag{10}$$

Formulae (10) lead to a solution for the pentagon equation, see (Kashaev, Sergeev 1996). In fact, they can be obtained by considering the classical limit of the quantum dilogarithm (Sergeev, 1996).

Proposition 1 *S extends to a unique covariant functor from $\mathfrak{G}_{\tilde{\Delta}_\Sigma}$ into $\mathfrak{P}$.*

The mapping class group M_Σ acts in $\tilde{\Delta}_\Sigma$, and for each d.i.t. τ one has the induced mapping

$$m \colon \dot{\tau} \to m(\dot{\tau}), \quad \forall m \in M_\Sigma,$$

of the corresponding set of triangles. For each $m \in M_\Sigma$ and d.i.t. τ associate morphism

$$\mathfrak{s}_m(\tau) \colon S(\tau) \ni f \mapsto S(\tau \cdot m(\tau))(f \circ m^{-1}) \in S(\tau).$$

Proposition 2 *Mapping $m \mapsto \mathfrak{s}_m$ is a group homomorphism from M_Σ into functorial isomorphisms from S to S.*

For a path γ in triangle $t \in \dot{\tau}$, $\tau \in \tilde{\Delta}_\Sigma$, connecting interior points of i.a. a and b with the initial point being at i.a. a, assign function

$$\mathfrak{u}(\gamma) := \begin{cases} 1/t_1 & a = 1^t, \ b = 2^t; \\ t_2 & a = 0^t, \ b = 1^t; \\ t_2/t_1 & a = 0^t, \ b = 2^t; \end{cases} \qquad \mathfrak{u}(\gamma^{-1}) := 1/\mathfrak{u}(\gamma). \tag{11}$$

Let γ be an element of the first integer homology group of Σ represented by an oriented loop γ in Σ (we shall not distinguish between loops and the homology classes they define), which consecutively intersects i.a. $c_1, \ldots, c_n$. For each

$i = 1, \ldots, n$ the segment γ_i between i.a. c_i and c_{i+1} ($c_{n+1} = c_1$) is contained in triangle $t_i \in \dot{\tau}$. So γ can be represented as a composition of paths

$$\gamma = \gamma_n \cdots \gamma_2 \gamma_1,$$

where definition (11) for the each factor makes sense. Consider a sum

$$\langle \mu_\tau, \gamma \rangle := \sum_{i=1}^{n} \ln \mathfrak{u}(\gamma_i) \tag{12}$$

which will be called *holonomy* aroung γ. It is easy to see that this quantity does depend on the homology class of γ only, and it is linear in the second argument

$$\langle \mu_\tau, \gamma_1 \gamma_2 \rangle = \langle \mu_\tau, \gamma_1 \rangle + \langle \mu_\tau, \gamma_2 \rangle.$$

So, it defines *momentum mapping*

$$\mu_\tau \colon S(\tau) \to H^1(\Sigma, \mathbb{R}), \quad \langle \mu_\tau(f), \gamma \rangle := \langle \mu_\tau, \gamma \rangle(f), \quad \gamma \in H_1(\Sigma, \mathbb{R}), \tag{13}$$

which is in fact a group homomorphism.

Proposition 3 *The Poisson bracket, associated with symplectic structure (7), of holonomies around two loops γ_1 and γ_2 is given by their intersection index: $\{\langle \mu_\tau, \gamma_1 \rangle, \langle \mu_\tau, \gamma_2 \rangle\} = \gamma_1 \circ \gamma_2$.*

Define a group homomorphism

$$P\mathbb{R}_+^V \ni f \mapsto \xi_f := \sum_{v \in V} \gamma_v \ln f(v) \in H_1(\Sigma, \mathbb{R}), \tag{14}$$

where γ_v is a small loop, encircling puncture v in the counterclockwise direction w.r.t. the orientation of Σ.

Proposition 4 *The exponential mapping of the Hamiltonian vector field, corresponding to function $\langle \mu_\tau, \xi_f \rangle$, coincides with the group action (6).*

Define a set of group homomorphisms

$$\mathfrak{f}(\tau) \colon R(E(\tau)) \to S(\tau), \quad \mathfrak{f}(\tau)^*(\mathbf{t}) := (2^t/1^t, 0^t/1^t), \quad \forall \tau \in \tilde{\Delta}_\Sigma. \tag{15}$$

Proposition 5 $\mathfrak{f}$ *is a functorial morphism from $R \circ E$ to S such that the following sequence of group homomorphisms is exact*

$$1 \to \mathbb{R}_+ \xrightarrow{\iota} R \circ E(\tau) \xrightarrow{\mathfrak{f}(\tau)} S(\tau) \xrightarrow{\mu_\tau} H^1(\Sigma, \mathbb{R}) \to 0, \quad \forall \tau \in \tilde{\Delta}_\Sigma,$$

where $\iota(a)(c) := a$, $a \in \mathbb{R}_+$, $c \in E(\tau)$, and

$$\mathfrak{f}(\tau) \circ \mathfrak{r}_m(E(\tau)) = \mathfrak{s}_m(\tau) \circ \mathfrak{f}(\tau), \quad \forall m \in M_\Sigma, \quad \forall \tau \in \tilde{\Delta}_\Sigma.$$

To summarize, the Teichmüller space $\mathcal{T}_\Sigma$ with the Weil–Petersson symplectic structure and the action of the mapping class group M_Σ can be described as the Hamiltonian reduction of $S(\tau)$ w.r.t. $P\mathbb{R}_+^V$ over the zero value of the momentum mapping (13):

$$\mathcal{T}_\Sigma \simeq \mathbb{R}_+^{E(\tau)} / \mathbb{R}_+^V \simeq \mu_\tau^{-1}(0) / P\mathbb{R}_+^V, \quad \forall \tau \in \tilde{\Delta}_\Sigma. \tag{16}$$

2 Quantization

Denote by $\mathfrak{A}$ the category of unital associative algebras over $\mathbb{C}$. Fix a real positive number $\hbar$. With each d.i.t. τ associate object $A(\tau) \in \mathrm{Ob}\mathfrak{A}$, which is the skew-field of fractions of a skew polynomial algebra generated by elements

$$\{\hat{\mathbf{t}} := (\hat{t}_1, \hat{t}_2)\}_{t \in \dot{\tau}}$$

subject to the following relations,

$$\hat{t}_1 \hat{t}_2 = q^2 \hat{t}_2 \hat{t}_1, \quad \hat{t}_i \hat{t'}_j = \hat{t'}_j \hat{t}_i, \quad i, j = 1, 2, \ \forall t \neq t' \in \dot{\tau}, \quad q := \exp(\sqrt{-1}\hbar).$$

For d.i.t. τ and τ_t, connected by the elementary change of decoration in triangle t, see Fig. 2, associate morphism

$$A(\tau_t \cdot \tau) \colon A(\tau_t) \to A(\tau), \quad A(\tau_t \cdot \tau)(\hat{\mathbf{x}}) := \begin{cases} (q\hat{t}_1^{-1}\hat{t}_2, \hat{t}_1^{-1}) & \text{if } x = t'; \\ \hat{\mathbf{x}} & \text{otherwise;} \end{cases} \tag{17}$$

For d.i.t. τ and τ^e, connected by the decorated elementary move along i.a. e depicted in Fig. 3, associate morphism

$$A(\tau^e \cdot \tau) \colon A(\tau^e) \to A(\tau), \quad A(\tau^e \cdot \tau)(\hat{\mathbf{t}}) = \begin{cases} \hat{\mathbf{x}} \bullet \hat{\mathbf{y}} & t = x'; \\ \hat{\mathbf{x}} * \hat{\mathbf{y}} & t = y'; \\ \hat{\mathbf{t}} & \text{otherwise;} \end{cases} \tag{18}$$

where notation (10) is used.

Proposition 6 *A extends uniquely to a contravariant functor from $\mathfrak{G}_{\tilde{\Delta}_\Sigma}$ into $\mathfrak{A}$.*

Each $m \in M_\Sigma$ induces algebra isomorphism

$$m \colon A(\tau) \ni \hat{\mathbf{t}} \mapsto \hat{\mathbf{t}}' \in A(m(\tau)), \quad t' = m(t) \in m(\dot{\tau}), \quad \forall \tau \in \tilde{\Delta}_\Sigma.$$

Define

$$\mathfrak{a}_m(\tau) := A(m(\tau) \cdot \tau) \circ m \colon A(\tau) \to A(\tau), \quad \forall m \in M_\Sigma, \quad \forall \tau \in \tilde{\Delta}_\Sigma$$

Proposition 7 *Mapping $m \mapsto \mathfrak{a}_m$ is a group homomorphism from M_Σ to functorial automorphisms of A.*

Thus, representations of algebra $A(\tau)$ by linear operators in Hilbert spaces should lead to (projective) representations of the mapping class group M_Σ.

2.1 Representations

2.1.1 Non-compact representation

The $*$-algebra structure in $A(\tau)$ defined by

$$\hat{t}_i^* = \hat{t}_i, \tag{19}$$

is natural from the viewpoint of the underlying classical phase space. Consider

$$B(\tau) := L^2(\mathbb{R}_+^{\dot{\tau}}, d\mu(\tau) := \prod_{t\in\dot{\tau}} d\ln t), \quad \forall \tau \in \tilde{\Delta}_\Sigma,$$

the Hilbert space of square integrable functions on $\mathbb{R}_+^{\dot{\tau}}$ w.r.t. the measure $d\mu(\tau)$. Algebra $A(\tau)$ is realized linearly in $B(\tau)$ through the formulae

$$\mathfrak{b}(\hat{t}_1)f := ft, \quad \mathfrak{b}(\hat{t}_2)f := \exp(-2\sqrt{-1}\hbar t\partial/\partial t)f, \quad \forall f \in B(\tau).$$

Proposition 8 *Let function $\psi(z)$ be a solution of the functional equation*

$$\psi(z - \sqrt{-1}\hbar) = \psi(z + \sqrt{-1}\hbar)(1 + \exp z), \quad z \in \mathbb{C}, \tag{20}$$

then operator

$$T_{x,y} := y^{-x\partial/\partial x}\psi(\ln x + 2\hbar\sqrt{-1}(x\partial/\partial x - y\partial/\partial y))$$

satisfies equations

$$T_{x,y}\mathfrak{b}(\hat{\mathbf{x}} \bullet \hat{\mathbf{y}}) = \mathfrak{b}(\hat{\mathbf{x}})T_{x,y}, \quad T_{x,y}\mathfrak{b}(\hat{\mathbf{x}} * \hat{\mathbf{y}}) = \mathfrak{b}(\hat{\mathbf{y}})T_{x,y}.$$

A particular solution to eqn (20) is given by the formula (Faddeev, 1995)

$$\psi(z) := \exp \frac{1}{4}\int_{-\infty}^{+\infty} \frac{\exp(-\sqrt{-1}xz)}{\sinh(\pi x)\sinh(\hbar x)}\frac{dx}{x},$$

where the singularity of the integrand at $x = 0$ is put below the contour of integration. In this case operator $T_{x,y}$ is unitary.

2.1.2 Compact representation

Let $N \geq 2$ be a positive integer, and ω, a complex primitive N^{th} root of unity. With any d.i.t. τ associate object

$$C(\tau) := \{f\colon S(\tau) \to M(\tau)\} \in \mathrm{Ob}\mathfrak{A},$$

where unital finite dimensional algebra $M(\tau)$ over $\mathbb{C}$ is generated by elements

$$\{\check{\mathbf{t}} := (\check{t}_1, \check{t}_2)\}_{t\in\dot{\tau}}$$

subject to the following relations,

$$\check{t}_1\check{t}_2 = \omega\check{t}_2\check{t}_1, \quad \check{t}_i\check{t}'_j = \check{t}'_j\check{t}_i, \quad \check{t}_i^N = 1, \quad i,j = 1,2,\ \forall t \neq t' \in \dot{\tau};$$

the algebra structure in $C(\tau)$ being given by the pointwise algebraic operations in $M(\tau)$. Note that algebra $M(\tau)$ has a unique up to isomorphism irreducible finite dimensional representation with the $*$-algebra structure given by

$$\check{t}_i^* = \check{t}_i^{-1}, \quad i = 1,2, \quad \forall t \in \dot{\tau}.$$

Algebra $A(\tau)$ at $q^2 = \omega$ is represented in $C(\tau)$ through the formulae

$$\mathfrak{c}(\tau)\colon A(\tau) \to C(\tau), \quad \mathfrak{c}(\tau)(\hat{t}_i)(h) := \check{t}_i \sqrt[N]{t_i(h)}, \quad i = 1,2,\ \forall h \in S(\tau),$$

where positive N^{th} roots are taken. This repsentation does not respect the $*$-structure (19).

For d.i.t. τ and τ', where $\tau' = \tau_t$ (elementary change of decoration in triangle t) or $\tau' = \tau^e$ (elementary move along i.a. e), associate morphisms

$$C(\tau' \cdot \tau)\colon C(\tau') \to C(\tau),$$

$$C(\tau' \cdot \tau)(f)(h) := M_h(\tau' \cdot \tau)(f(S(\tau' \cdot \tau)(h))), \quad \forall f \in C(\tau'), \quad \forall h \in S(\tau),$$

where algebra isomorphisms

$$M_h(\tau' \cdot \tau)\colon M(\tau') \to M(\tau), \quad \forall h \in S(\tau)$$

are defined by

$$M_h(\tau_t \cdot \tau)(\check{\mathbf{x}}) := \begin{cases} (\omega^{1/2}\check{t}_1^{-1}\check{t}_2, \check{t}_1^{-1}) & \text{if } x = t'; \\ \check{\mathbf{x}} & \text{otherwise,} \end{cases} \quad \omega^{N/2} = (-1)^{N-1};$$

$$M_h(\tau^e \cdot \tau)(\check{\mathbf{t}}) := \begin{cases} \check{\mathbf{x}} \bullet_h \check{\mathbf{y}} & t = x'; \\ \check{\mathbf{x}} *_h \check{\mathbf{y}} & t = y'; \\ \check{\mathbf{t}} & \text{otherwise;} \end{cases}$$

$$\check{\mathbf{x}} \bullet_h \check{\mathbf{y}} := \left(\check{x}_1\check{y}_1, (\check{x}_2 + \check{x}_1\check{y}_2 h_{x,y})/\sqrt[N]{1 + h_{x,y}^N}\right), \quad h_{x,y} := \sqrt[N]{x_1y_2/x_2}(h),$$

$$\check{\mathbf{x}} *_h \check{\mathbf{y}} := (\check{y}_1\check{x}_2(\check{\mathbf{x}} \bullet_h \check{\mathbf{y}})_2^{-1}, \check{y}_2(\check{\mathbf{x}} \bullet_h \check{\mathbf{y}})_2^{-1}),$$

with positive N^{th} roots being chosen; see also Figs 2, 3 for the notation of the triangles.

Proposition 9 *C extends uniquely to a contravariant functor from $\mathfrak{G}_{\tilde{\Delta}_\Sigma}$ into $\mathfrak{A}$, $\mathfrak{c}$ being a functorial morphism from A to C.*

Proposition 10 *Let $\Psi_{\omega,\lambda}(w)$ be a solution of the functional equation*

$$\Psi_\lambda(\omega w)(1 - w\lambda)/\lambda' = \Psi_\lambda(w), \tag{21}$$

where

$$\lambda, \lambda' \in \mathbb{R}_+,\ \lambda' = \sqrt[N]{1 + \lambda^N}, \quad w^N = -1;$$

then the element

$$T_{h,x,y} := (\sum_{i,j=0}^{N-1} \omega^{-ij} \check{y}_1^i \check{x}_2^j) \Psi_{h_{x,y}}(-\check{x}_2^{-1} \check{x}_1 \check{y}_2),$$

satisfies the equations

$$T_{h,x,y} \check{\mathbf{x}} \bullet_h \check{\mathbf{y}} = \check{\mathbf{x}} T_{h,x,y}, \quad T_{h,x,y} \check{\mathbf{x}} *_h \check{\mathbf{y}} = \check{\mathbf{y}} T_{h,x,y}.$$

Solution of the functional equation (21) is unique up to a complex factor. It satisfies the pentagon equation (Faddeev, Kashaev 1992) which is a non-commutative analogue of Roger's five-term identity for the dilogarithm.

Summary

The Teichmüller space of marked conformal types of hyperbolic metrics on a punctured surface with the Weil–Petersson symplectic structure and the action of the mapping class group can be described as the Hamiltonian reduction (16) of a finite dimensional symplectic manifold (5). The quantization of the latter is straightforward, and the action of the mapping class group is realized through the quantum dilogarithms. These results relate the quantum hyperbolic invariant of knots (Kashaev 1994, 1995, 1997) to the quantum theory of Teichmüller spaces of punctured surfaces.

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